Elliptic solutions of the quintic complex one-dimensional Ginzburg–Landau equation

2007 J. Phys. A: Math. Theor. 40 9833

(http://iopscience.iop.org/1751-8121/40/32/009)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9833–9844 doi:10.1088/1751-8113/40/32/009

Elliptic solutions of the quintic complex one-dimensional Ginzburg–Landau equation

S Yu Vernov

Skobeltsyn Institute of Nuclear Physics, Moscow State University, Vorobyevy Gory, Moscow 119992, Russia

E-mail: svernov@theory.sinp.msu.ru

Received 27 February 2007, in final form 26 June 2007
Published 24 July 2007
Online at stacks.iop.org/JPhysA/40/9833

Abstract

The Conte–Musette method has been modified for the search of only elliptic solutions to systems of differential equations. A key idea of this *a prior* restriction is to simplify calculations by means of the use of a few Laurent-series solutions instead of one and the use of the residue theorem. The application of this approach to the quintic complex one-dimensional Ginzburg–Landau equation (CGLE5) allows us to find elliptic solutions in the wave form. Restrictions on coefficients, which are necessary conditions for the existence of elliptic solutions for the CGLE5, have been found as well. We demonstrate that to find elliptic solutions the analysis of a system of differential equations is preferable to the analysis of the equivalent single differential equation.

PACS numbers: 04.20.Jb, 02.30.−f, 47.27.ed

1. Introduction

At present methods for construction of special solutions of nonintegrable systems in terms of elementary (more precisely, degenerated elliptic) and elliptic functions are actively developed [1–20] (see also [21] and references therein). Some of these methods are intended for the search for elliptic solutions only [11, 15]; others allow us to find either solutions in terms of elementary functions only [2–4, 20] or both types of solutions [1, 5–10, 12–14, 16–19]. Note that the methods [13, 17] allow us to find multivalued solutions as well.

A nonintegrable system has elliptic solutions, only if it has the Laurent-series solutions. Such local solutions can be constructed by means of the Ablowitz–Ramani–Segur algorithm of the Painlevé test [22] (see also [21, 23, 24]). In this way one obtains solutions only as formal series. That is sufficient because really only a finite number of coefficients of these series are used. Examples of construction of such solutions are given in [4, 25]. The Laurent-series solutions give the information about the global behaviour of differential systems and assist to look for both exact solutions [5] and the first integrals [26]. The Laurent-series solutions can be used to prove the nonexistence of elliptic solutions [27, 28] as well.

In [5], a new method for construction of single-valued special solutions for nonintegrable differential equations has been proposed. A key idea of this method is the use of the Laurent-series solutions to transform the initial differential equation into a nonlinear system of algebraic equations. Using this method one can in principle find all elliptic and degenerate elliptic solutions. Unfortunately if the initial differential equation includes a large number of numeric parameters, then it is difficult to solve the obtained nonlinear system of algebraic equations.

The goal of this paper is to propose a modification of the Conte–Musette method [5], which allows us to seek elliptic solutions only. We show that in this case it is possible to fix some parameters of the initial differential system and therefore simplify the resulting system of algebraic equations. To do this we use the Hone method, which has been proposed to prove the non-existence of elliptic solutions [27]. Note that using our approach one can find in principal all elliptic solutions.

In [5], the Musette and Conte have transformed the initial system of two coupled ordinary differential equations into the equivalent single differential equation and only after this have constructed the Laurent-series solutions. In our paper we demonstrate that the analysis of the system of differential equations can give more information than the analysis of the equivalent differential equation.

2. The Conte–Musette method for the system of differential equations

In [5], Conte and Musette have proposed a way to search for elliptic and degenerate elliptic solutions to a polynomial autonomous differential equation. In this section we reformulate this method for a system of such equations:

$$F_i\left(\vec{\mathbf{y}}_{;t}^{(n)}, \vec{\mathbf{y}}_{;t}^{(n-1)}, \dots, \vec{\mathbf{y}}_{;t}, \vec{\mathbf{y}}\right) = 0, \qquad i = 1, \dots, N, \tag{1}$$

where $\vec{\mathbf{y}} = \{y_1(t), y_2(t), \dots, y_L(t)\}$ and $y_{j;t}^{(k)} = \frac{\mathrm{d}^k y_j}{\mathrm{d}t^k}$.

It is known that any elliptic function (including any degenerate one) is a solution of some first-order polynomial autonomous differential equation. An elliptic function, whose poles have the order p, should satisfy the following equation (see the details in [5]):

$$\sum_{k=0}^{m} \sum_{j=0}^{(p+1)(m-k)/p} h_{j,k} y^j {y_t}^k = 0, \qquad h_{0,m} = 1, \tag{2}$$

where m is some positive integer number. The summation in (2) runs over nonnegative integers j that are less than or equal to $(p+1)(m-k)/p$. Coefficients $h_{j,k}$ are constants. It is proven that the general solution of (2) is either an elliptic function, or a rational function of $\mathrm{e}^{\gamma x}$, γ being some constant, or a rational function of x. Note that the third case is a degeneracy of the second one, which in turn is a degenerate case of the first one.

The Conte–Musette algorithm is the following [5] (see also [6]):

(i) Choose a positive integer number m, define the form of (2) and calculate the number of unknown coefficients $h_{j,k}$.

(ii) Construct solutions of system (1) in the form of Laurent series. System (1) is autonomous, so the coefficients of the Laurent series do not depend on the position of the singular point. They may depend on values of the numerical parameters of (1). In addition, some of these coefficients (the number of which is less than the order of system (1)) may take arbitrary values and have to be considered as new numerical parameters. One should compute more coefficients of the Laurent series than the number of numerical parameters in the Laurent series plus the number of coefficients $h_{j,k}$.

(iii) Choose a Laurent-series expansion for some function y_k and substitute the obtained Laurent-series coefficients into (2). This substitution transforms (2) into a linear and overdetermined system in $h_{j,k}$ with coefficients depending on numerical parameters.

(iv) Eliminate coefficients $h_{j,k}$ and get a nonlinear system in the parameters.

(v) Solve the obtained nonlinear system.

Conte and Musette note that a computer algebra package is highly recommended for using their method [6]. For the given system it is easy to calculate the Laurent-series solutions to any accuracy. These computations base on the Painlevé test, which has been implemented in the most popular computer algebra systems [29–32]. The first package of computer algebra procedures, which realize the third and the fourth steps of the algorithm, has been written in AMP [33] by Conte. One can also use our Maple and REDUCE packages of procedures, which are accessible via the Internet [34] and are described in [35, 36]. So, one passes the first four steps of algorithm without any difficulties.

At the fifth (last) step one should solve an overdetermined system of nonlinear algebraic equations. The standard method for solving of such systems is the construction of a lexicographically ordered Gröbner basis [37] (see also [38–40]). This construction is not simple even with the use of modern computer algebra systems. The computer memory required for constructing of this basis depends, in the general case, exponentially on the number of the unknowns. Therefore, this number should be made as small as possible.

The purpose of this paper is to show that we can essentially simplify the algebraic system of equations, which we have to solve on the last step of the Conte–Musette method, if we restrict ourselves to search for only elliptic solutions.

In [5], Conte and Musette have used their method to find wave solutions of the complex cubic Ginzburg–Landau equation (CGLE3). The nonexistence of elliptic travelling and standing wave solutions of the CGLE3 has been proved in [27] and [28] respectively. In section 4 we seek elliptic solutions of the quintic complex Ginzburg–Landau equation (CGLE5) using our modification of the Conte–Musette method. Note that both the CGLE3 and the CGLE5 have only one-parameter Laurent-series solutions in the wave form and there exist only a finite number of such solutions.

Our approach can be effectively used in investigation of any system (1) or a single differential equation, which has only a finite number of different Laurent-series solutions. Note that for a wide class of such differential equations it has been proved that all their meromorphic solutions are elliptic or degenerated elliptic functions [41].

3. The quintic complex Ginzburg–Landau equation

The one-dimensional quintic complex Ginzburg–Landau equation (CGLE5) is a generalization of the one-dimensional cubic complex Ginzburg–Landau equation [42] (CGLE3), which is one of the most-studied nonlinear equations (see [43] and references therein). Moreover, the CGLE5 is a generic equation which describes many physical phenomena, for example, the behaviour of travelling patterns in binary fluid convection [44] and the large-scale behaviour of many nonequilibrium pattern-forming systems [45].

The CGLE5 is as follows:

$$\mathrm{i}\mathcal{A}_t + p\mathcal{A}_{xx} + q|\mathcal{A}|^2\mathcal{A} + r|\mathcal{A}|^4\mathcal{A} - \mathrm{i}\gamma\mathcal{A} = 0, \tag{3}$$

where the subscribes denote partial derivatives, $\mathcal{A}_t \equiv \frac{\partial\mathcal{A}}{\partial t}$, $\mathcal{A}_{xx} \equiv \frac{\partial^2\mathcal{A}}{\partial x^2}$, $p, q, r \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

One of the most important directions in the study of the CGLE5 is the consideration of its travelling wave reduction [1, 4, 6, 44, 46, 47],

$$\mathcal{A}(x,t) = \sqrt{M(\xi)}\,\mathrm{e}^{\mathrm{i}(\varphi(\xi)-\omega t)}, \qquad \xi = x - ct, \qquad c \in \mathbb{R}, \qquad \omega \in \mathbb{R}. \tag{4}$$

Substituting (4) into (3) and multiplying both sides of this equation on $4M^2/A$ we obtain

$$\begin{aligned}2pM''M - pM'^2 + 4\mathrm{i}p\psi MM' + 2(2\omega - \mathrm{i}c - 2\mathrm{i}\gamma \\ + 2c\psi - 2p\psi^2 + 2\mathrm{i}p\psi')M^2 + 4qM^3 + 4rM^4 = 0,\end{aligned} \tag{5}$$

where $\psi \equiv \varphi' \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}$, $M' \equiv \frac{\mathrm{d}M}{\mathrm{d}\xi}$. Equation (5) is a system of two equations: both real and imaginary parts of its left-hand side have to be equal to zero. Dividing (5) by p and separating real terms from imaginary ones, we obtain the following system:

$$\begin{cases} 2MM'' - M'^2 - 4M^2\tilde{\psi}^2 - 2\tilde{c}MM' + 4g_iM^2 + 4d_rM^3 + 4u_rM^4 = 0, \\ M\tilde{\psi}' + \tilde{\psi}(M' - \tilde{c}M) - g_rM + d_iM^2 + u_iM^3 = 0, \end{cases} \tag{6}$$

where the new real variables are as follows:

$$u_r + \mathrm{i}u_i = \frac{r}{p}, \qquad d_r + \mathrm{i}d_i = \frac{q}{p}, \qquad s_r - \mathrm{i}s_i = \frac{1}{p}, \tag{7}$$

$$g_r + \mathrm{i}g_i = (\gamma + \mathrm{i}\omega)(s_r - \mathrm{i}s_i) + \frac{1}{2}c^2s_is_r + \frac{\mathrm{i}}{4}c^2s_r^2 \tag{8}$$

and

$$\tilde{\psi} \equiv \psi - \frac{cs_r}{2}, \qquad \tilde{c} \equiv cs_i. \tag{9}$$

System (6) includes seven real parameters: g_r, g_i, d_r, d_i, u_r, u_i and $\tilde{c}$. Note that to obtain (6) from (5) we assume that the functions $M(\xi)$ and $\psi(\xi)$ are real.

The standard way to construct exact solutions for system (6) is to transform it into the equivalent third-order differential equation for M. We rewrite the first equation of system (6) as

$$\tilde{\psi}^2 = \frac{G}{M^2}, \tag{10}$$

where

$$G \equiv \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{\tilde{c}}{2}MM' + g_iM^2 + d_rM^3 + u_rM^4. \tag{11}$$

From (10) it follows that

$$\tilde{\psi}'\tilde{\psi} = \frac{G'M - 2GM'}{2M^3}. \tag{12}$$

Multiplying the second equation of (6) on $\tilde{\psi}$ and substituting (10) and (12) into it, we express $\tilde{\psi}$ in terms of M and its derivatives,

$$\tilde{\psi} = \frac{G' - 2\tilde{c}G}{2M^2(g_r - d_iM - u_iM^2)} \tag{13}$$

and obtain the third-order equation for M,

$$(G' - 2\tilde{c}G)^2 + 4GM^2(g_r - d_iM - u_iM^2)^2 = 0. \tag{14}$$

4. Construction of elliptic solutions

4.1. The Laurent-series solutions

We consider the case $(p/r) \notin \mathbb{R}$, which corresponds to the condition $u_i \neq 0$. In this case (14) is not integrable [4] and its general solution (which should depend on three arbitrary integration constants) is not known. Using the Painlevé analysis [4] it has been shown that single-valued solutions of (6) can depend on only one arbitrary parameter. System (6) is autonomous, so this parameter is ξ_0: if $M = f(\xi)$ is a solution, then $M = f(\xi - \xi_0)$, where $\xi_0 \in \mathbb{C}$ has to be a solution. Special solutions in terms of elementary functions have been found in [1, 4, 6, 47]. All known exact solutions of (6) are elementary (rational or hyperbolic) functions. The full list of these solutions is presented in [6]. Note that in the case $u_i = 0$ elliptic solutions have been found [48]. The purpose of this section is to find at least one elliptic solution of (6) with $u_i \neq 0$.

System (6) is invariant under the transformation

$$\tilde{\psi} \to -\tilde{\psi}, \qquad g_r \to -g_r, \qquad d_i \to -d_i, \qquad u_i \to -u_i; \tag{15}$$

therefore, we can assume that $u_i > 0$ without loss of generality. Moreover, using scale transformations

$$M \to \lambda M, \qquad d_r \to \frac{d_r}{\lambda}, \quad d_i \to \frac{d_i}{\lambda}, \qquad u_r \to \frac{u_r}{\lambda^2}, \qquad u_i \to \frac{u_i}{\lambda^2}, \tag{16}$$

we can always put $u_i = 1$.

Let us construct Laurent-series solutions to system (6). We assume that in a sufficiently small neighbourhood of the singularity point ξ_0 the functions $\tilde{\psi}$ and M tend to infinity as some powers of $\xi - \xi_0$,

$$\tilde{\psi} = A(\xi - \xi_0)^{\alpha} + \cdots \qquad \text{and} \qquad M = B(\xi - \xi_0)^{\beta} + \cdots, \tag{17}$$

where α and β are negative integer numbers and, of course, $A \neq 0$ and $B \neq 0$. Substituting (17) into (6) we obtain that two or more terms in the equations of system (6) balance if and only if $\alpha = -1$ and $\beta = -1$. In other words, in this case these terms have equal powers and the other terms can be ignored as $\xi \longrightarrow \xi_0$. From the system

$$\begin{cases} B^2(3 - 4A^2 + 4u_r B^2) = 0, \\ 2A - B^2 = 0, \end{cases} \tag{18}$$

we obtain four nonzero solutions:

$$A_1 = u_r + \tfrac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_1 = \sqrt{2u_r + \sqrt{4u_r^2 + 3}}, \tag{19}$$

$$A_2 = u_r + \tfrac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_2 = -\sqrt{2u_r + \sqrt{4u_r^2 + 3}}, \tag{20}$$

$$A_3 = u_r - \tfrac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_3 = \sqrt{2u_r - \sqrt{4u_r^2 + 3}}, \tag{21}$$

$$A_4 = u_r - \tfrac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_4 = -\sqrt{2u_r - \sqrt{4u_r^2 + 3}}. \tag{22}$$

Therefore, system (6) has four types of Laurent-series solutions. Denote them as follows:

$$\tilde{\psi}_k = \frac{A_k}{\xi} + a_{k,0} + a_{k,1}\xi + \cdots, \qquad M_k = \frac{B_k}{\xi} + b_{k,0} + b_{k,1}\xi + \cdots, \qquad k = 1, \ldots, 4. \tag{23}$$

4.2. Restrictions on values of parameters

The purpose of this subsection is to check the possibility of existence of elliptic solutions for (6). To do this we use the Hone method [27] based on the residues theorem (we recall in the appendix the statement of the residues theorem and other basic properties of elliptic functions). To make the Hone method more effective we propose two modifications.

First, we use the Laurent-series expansions not only of $M(\xi)$, but also of $\psi(\xi)$, which should be an elliptic function as well. Indeed, let $M(\xi)$ be a nontrivial elliptic function. If $\tilde{\psi}$ is a constant, then from the second equation of system (6) it follows that $M(\xi)$ cannot be a nontrivial elliptic function. Therefore, using (13), we conclude that $\tilde{\psi}(\xi)$ has to be a nontrivial elliptic function.

Second we prove and use the following lemma.

Lemma 1. *An elliptic function cannot have two poles with the same Laurent-series expansions in its fundamental parallelogram of periods.*

Proof. Let some elliptic function $\varrho(\xi)$ have two poles in points ξ_0 and ξ_1, which belong to the fundamental parallelogram of periods. The corresponding Laurent series are the same and have the convergence radius R. Then the function $\upsilon(\xi) = \varrho(\xi - \xi_0) - \varrho(\xi - \xi_1)$ is an elliptic function as a difference between two elliptic functions with the same periods. At the same time for all ξ such that $|\xi| < R$ $\upsilon(\xi) = 0$; therefore, $\upsilon(\xi) \equiv 0$ and $\varrho(\xi - \xi_0) \equiv \varrho(\xi - \xi_1)$ and $\xi_1 - \xi_0$ is a period of $\varrho(\xi)$. It contradicts to our assumption that both points ξ_0 and ξ_1 belong to the fundamental parallelogram of periods. □

Let us consider the fundamental parallelogram of periods for the function $M(\xi)$ and define a number of its poles in this domain. Let M have a pole of type M_1; hence, according to the residue theorem, it should have a pole of type M_2 (not a pole of type M_4 because u_r is a real parameter). So $\tilde{\psi}$ has poles with the Laurent series $\tilde{\psi}_1$ and $\tilde{\psi}_2$. As an elliptic function it should have a pole of type $\tilde{\psi}_3$ or $\tilde{\psi}_4$ as well. It means that the function $M(\xi)$ should have a pole of type M_3 and hence a pole of type M_4. So $M(\xi)$ should have at least four different poles in its fundamental parallelogram of periods. Using lemma 1, we obtain that the function $M(\xi)$ cannot have the same poles in the fundamental parallelogram of periods. Therefore, $M(\xi)$ has exactly four poles in its fundamental parallelogram of periods. In this case by means of the residue theorem for $\tilde{\psi}$ we obtain

$$u_r = 0. \tag{24}$$

We obtain that the CGLE5 with $u_r \neq 0$ has no elliptic solution in the wave form. In the case $u_r = 0$ possible elliptic solutions should have four simple poles in the fundamental parallelogram of periods, and, therefore, have the following form [49]:

$$M(\xi - \xi_0) = C + \sum_{k=1}^{4} B_k \zeta(\xi - \xi_k), \tag{25}$$

where the function $\zeta(\xi) = -\int \wp(\xi)\,\mathrm{d}\xi$. Constants B_k are given by (19)–(22). To define the function M one should find periods of the Weierstrass elliptic function $\wp(\xi)$ and the constants C and ξ_k.

To obtain restrictions on other parameters, we apply the residue theorem to the functions $\tilde{\psi}^2$, $\tilde{\psi}^3$ and so on. The residue theorem for the function $\tilde{\psi}^2$ gives the equation

$$\sum_{k=1}^{4} A_k a_{k,0} = 0. \tag{26}$$

The values of $a_{k,0}$ are as follows ($u_r = 0$):

$$a_{1,0} = \frac{\sqrt{3}}{48}(6\tilde{c} - \sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r), \tag{27}$$

$$a_{2,0} = \frac{\sqrt{3}}{48}(6\tilde{c} + \sqrt[4]{27}d_i + 15\sqrt[4]{3}d_r), \tag{28}$$

$$a_{3,0} = -\frac{\sqrt{3}}{48}(6\tilde{c} + \mathrm{i}(\sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r)), \tag{29}$$

$$a_{4,0} = -\frac{\sqrt{3}}{48}(6\tilde{c} - \mathrm{i}(\sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r)). \tag{30}$$

Substituting A_k and $a_{k,0}$ into (26), we obtain

$$\sum_{k=1}^{4} A_k a_{k,0} = \frac{3}{4}\tilde{c} = 0. \tag{31}$$

For the function $\tilde{\psi}^3$ the residue theorem gives

$$\sum_{k=1}^{4} A_k\left(A_k a_{k,1} + a_{k,0}^2\right) = 0. \tag{32}$$

Substituting the values of $a_{k,1}$ into (32) and using $\tilde{c} = 0$ we obtain

$$d_i{}^2 + 27d_r{}^2 = 0 \quad \rightarrow \quad d_i = \pm\mathrm{i}\sqrt{27}d_r. \tag{33}$$

The parameters d_r and d_i should be real; therefore, $d_r = 0$ and $d_i = 0$. This result means that $q = 0$ and the cubic term is absent in equation (3).

So, consideration of $\tilde{\psi}^2$ and $\tilde{\psi}^3$ gives three restrictions:

$$\tilde{c} = 0, \qquad d_r = 0 \qquad \text{and} \qquad d_i = 0. \tag{34}$$

The residue theorem for $\tilde{\psi}^4$ gives the restriction

$$g_i g_r = 0. \tag{35}$$

We have used the residue theorem for some other combinations of functions $\tilde{\psi}$, M and their derivatives, for example for $\tilde{\psi}^5$, $\tilde{\psi}^6$, $\tilde{\psi}\tilde{\psi}'$, $\tilde{\psi}M$, MM', $M^2, \ldots, M^6$, but do not obtain new restrictions on coefficients.

4.3. The use of the Conte–Musette algorithm

Taking into account (24) and (34) we obtain system (6) in the following form:

$$\begin{cases} 2MM'' - M'^2 - 4M^2\tilde{\psi}^2 + 4g_i M^2 = 0, \\ \tilde{\psi}'M + \tilde{\psi}M' - g_r M + M^3 = 0. \end{cases} \tag{36}$$

Multiplying the first and second equations of system (36) on M^2 and M respectively we obtain the following system:

$$\begin{cases} JJ'' - \frac{3}{4}J'^2 - 4J^2\tilde{\psi}^2 + 4g_i J^2 = 0, \\ \tilde{\psi}'J + \frac{1}{2}\tilde{\psi}J' - g_r J + J^2 = 0, \end{cases} \tag{37}$$

where $J = M^2$. It is easy to show that system (37) has only two different Laurent-series solutions.

To find elliptic solutions of system (36) we use the Conte–Musette method. The function $\tilde{\psi}(\xi)$ is a solution of both systems (36) and (37). Therefore it has only two different Laurent-series expansions, whereas the functions $M(\xi)$ should have four different Laurent-series expansions, so it is easier to find the first-order differential equation for $\tilde{\psi}(\xi)$ than $M(\xi)$.

Equation (2) with $m = 1$ has no elliptic solution. Let $\tilde{\psi}(\xi)$ satisfy (2) with $m = 2$,

$$\tilde{\psi}'^2 + (\tilde{h}_{2,1}\tilde{\psi}^2 + \tilde{h}_{1,1}\tilde{\psi} + \tilde{h}_{0,1})\tilde{\psi}' + \tilde{h}_{4,0}\tilde{\psi}^4 + \tilde{h}_{3,0}\tilde{\psi}^3 + \tilde{h}_{2,0}\tilde{\psi}^2 + \tilde{h}_{1,0}\tilde{\psi} + \tilde{h}_{0,0} = 0. \tag{38}$$

Substituting into (38) the Laurent series of $\tilde{\psi}$, which begins from A_1 (more exactly we use the first ten coefficients), we obtain the following solution $\tilde{h}_{k,j}$ for an arbitrary value of the parameter $g_r \neq 0$ and $g_i = 0$:

$$\tilde{h}_{4,0} = -\frac{4}{3}, \qquad \tilde{h}_{0,0} = -\frac{g_r^2}{9}, \qquad \tilde{h}_{3,0} = \tilde{h}_{2,0} = \tilde{h}_{1,0} = \tilde{h}_{0,1} = \tilde{h}_{1,1} = \tilde{h}_{2,1} = 0, \tag{39}$$

a few solutions with $g_i = 0$ and $g_r = 0$ and no solution for $g_i \neq 0$.

The straightforward substitution of the functions

$$\breve{\psi} = \frac{\sqrt{3}}{2\xi}, \qquad \breve{M} = \pm\frac{\sqrt[4]{3}}{\xi} \qquad \text{and} \qquad \hat{\psi} = -\frac{\sqrt{3}}{2\xi}, \qquad \hat{M} = \pm\mathrm{i}\frac{\sqrt[4]{3}}{\xi} \tag{40}$$

into (36) with $g_r = 0$ and $g_i = 0$ proves that they are exact solutions. The coefficients of the Laurent-series solutions do not include arbitrary parameters, so the obtained solutions are unique single-valued solutions and system (36) has no elliptic solution at $g_r = 0$ and $g_i = 0$.

In the case of solutions (39) the function $\tilde{\psi}(\xi)$ satisfies the equation

$$\tilde{\psi}'^2 = \frac{4}{3}\tilde{\psi}^4 + \frac{g_r^2}{9}. \tag{41}$$

The polynomial on the right-hand side of (41) has four different roots; therefore, $\tilde{\psi}$ is a non-degenerate elliptic function [49].

Surely we do not yet prove the existence of elliptic solutions to the CGLE5. For rigorous proof we should find the function $M(\xi)$ and check that this function is a solution of (14). The function $M(\xi)$ in a parallelogram of periods has four different Laurent-series expansions, so we should choose the parameter m such that solutions of (2) have four poles in their fundamental parallelogram of periods. Substituting the expression B/t into equation (2) with $p = 1$, we obtain that B should be a root of the mth degree polynomial (we use the condition $B \neq 0$). Therefore the minimal possible value of m, which corresponds to four different nonzero values of B is equal to 4.

Equation (2) for $m = 4$ and $p = 1$ has the following form:

$$\begin{aligned} M'^4 &+ (h_{2,3}M^2 + h_{1,3}M + h_{0,3})M'^3 + (h_{4,2}M^4 + h_{3,2}M^3 + h_{2,2}M^2 + h_{1,2}M + h_{0,2})M'^2 \\ &+ (h_{6,1}M^6 + h_{5,1}M^5 + h_{4,1}M^4 + h_{3,1}M^3 + h_{2,1}M^2 + h_{1,1}M \\ &+ h_{0,1})M' + h_{8,0}M^8 + h_{7,0}M^7 + h_{6,0}M^6 + h_{5,0}M^5 \\ &+ h_{4,0}M^4 + h_{3,0}M^3 + h_{2,0}M^2 + h_{1,0}M + h_{0,0} = 0. \end{aligned} \tag{42}$$

Substituting the Laurent series M_k from (23), we transform the left-hand side of (42) into the Laurent series, which has to be equal to zero. Therefore, we obtain the algebraic system in $h_{i,j}$ and g_r. The first algebraic equation, which corresponds to $1/\xi^8$ is

$$B_k^4\big(h_{8,0}B_k^4 - h_{6,1}B_k^3 + h_{4,2}B_k^2 - h_{2,3}B_k + 1\big) = 0, \qquad k = 1, \ldots, 4, \tag{43}$$

where B_k is defined by (19)–(22). To find all elliptic and degenerate elliptic solutions one can use only one of B_k. We seek only elliptic solutions; hence, we demand that all B_k have to satisfy (43), so we can consider equation (43) as the following system:

$$\begin{cases} h_{8,0}B_1^4 - h_{6,1}B_1^3 + h_{4,2}B_1^2 - h_{2,3}B_1 + 1 = 0, \\ h_{8,0}B_2^4 - h_{6,1}B_2^3 + h_{4,2}B_2^2 - h_{2,3}B_2 + 1 = 0, \\ h_{8,0}B_3^4 - h_{6,1}B_3^3 + h_{4,2}B_3^2 - h_{2,3}B_3 + 1 = 0, \\ h_{8,0}B_4^4 - h_{6,1}B_4^3 + h_{4,2}B_4^2 - h_{2,3}B_4 + 1 = 0. \end{cases} \tag{44}$$

Using the explicit values of B_k from (19)–(22), we obtain that

$$h_{8,0} = -\tfrac{1}{3}, \qquad h_{4,2} = 0, \qquad h_{6,1} = 0, \qquad h_{2,3} = 0. \tag{45}$$

Taking into account (45), from other equations of the algebraic system, we obtain

$$h_{6,0} = \tfrac{4}{3} g_r, \qquad h_{4,0} = -\tfrac{16}{9} g_r^2, \qquad h_{2,0} = \tfrac{64}{81} g_r^3, \tag{46}$$

all other $h_{i,j}$ are equal to zero. So, the equation for M has the form

$$M'^4 = \tfrac{1}{81} M^2 (3M^2 - 4g_r)^3. \tag{47}$$

Note that equation (47) can be rewritten in terms of J:

$$J'^4 = \tfrac{16}{81} J^3 (3J - 4g_r)^3. \tag{48}$$

Equation (14) at $u_i = 1$, $u_r = 0$, $\tilde{c} = 0$, $d_r = 0$, $d_i = 0$ and $g_i = 0$ has the form

$$\tfrac{1}{4}(M''')^2 - (2MM'' - M'^2)(M^2 - g_r)^2 = 0. \tag{49}$$

We multiply (49) on M'^2 and use (47) to express all derivatives of $M(\xi)$ in terms of the function $M(\xi)$. Straightforward calculations show that any solution of (47) satisfies (49). So, we obtain elliptic wave solutions of the CGLE5. If $s_i \neq 0$ these solutions are the standing wave solutions; in the opposite case ($s_i = 0$) the solutions can have an arbitrary speed c.

Note that we obtain (14) from (5) using the condition that $M(\xi)$ is a real function. For $g_r < 0$ and any initial value of M we obtain real solutions. In the case $g_r > 0$ there exists the minimal possible initial value of M for which real solutions exist and only particular solutions of (47) are suitable elliptic solutions to the CGLE5. The function $M(\xi)$ has the form (25), the values of constants can be determined from (47).

Using the first equation of (36) and equation (47) we find that

$$\tilde{\psi}^4 = \frac{(3M^2 - 4g_r)(96M^4 - 3M^2 - 32M^2 g_r + 4g_r)^2}{1296M^2}. \tag{50}$$

Summing up we can conclude that our modification of the Conte–Musette method allows us to get two results: we obtain new elliptic wave solutions of the CGLE5 and prove that these solutions are unique elliptic solutions for the CGLE5 with $g_r \neq 0$.

From (35) it follows that elliptic solutions can exist if g_r or g_i is equal to zero. For all nonzero values of g_r and zero g_i we have found elliptic solutions. In the case $g_r = g_i = 0$ there is no elliptic solution. In the case of zero g_r and nonzero g_i we substitute the obtained Laurent-series solutions M_k into equation (2) with $m = 1, \ldots, 4$ and do obtain neither elliptic functions nor degenerate elliptic solutions. We hope that the more detailed analysis of this case allows us to find all elliptic solutions for the CGLE5.

5. Construction of elliptic solutions for nonintegrable systems

The approach, which we have considered in the previous section, is applicable to many nonintegrable systems. The best domain of its applicability is nonintegrable systems of autonomous nonlinear ordinary differential equations with so-called finiteness property [41]: there are only a finite number of Laurent series that satisfy the system.

We propose the following way to the search for elliptic solutions to system (1):

(i) Choose the function y_k which should be elliptic.
(ii) Check should other functions be elliptic or not.
(iii) Construct the formal Laurent-series solutions.
(iv) Using the residue theorem find domains of values of numeric parameters at which the solution y_k can be an elliptic function.

(v) Choose a positive integer m, define the form of equation (2) and calculate the number of unknown coefficients $h_{j,k}$.

(vi) Calculate the sufficient number of coefficients for all Laurent series of y_k and substitute the obtained coefficients into (2). This substitution transforms (2) into a linear and overdetermined system in $h_{j,k}$ with coefficients depending on numerical parameters.

(vii) Eliminate coefficients $h_{j,k}$ and get a nonlinear system in parameters.

(viii) Solve the obtained nonlinear system.

Key ideas of our modification of the Conte–Musette method are to restrict ourselves to the search of elliptic solutions only and to consider not one Laurent series, but all found Laurent series. We show that the use of the Conte–Musette method in combination with the Hone method is very effective to find elliptic solutions.

6. Conclusions and discussions

Let us compare our approach with other methods for the construction of special solutions of nonintegrable systems (see [6] as a review of these methods).

To find solutions for the CGLE5 the authors of papers [1, 4, 47] put some restrictions on dependence between $\tilde{\psi}$ and M. The use of Laurent solutions allows us to search $\tilde{\psi}$ without any restrictions and without eliminating M from system (6).

Without the use of the Laurent-series solutions it would be very difficult to find the elliptic solutions of the CGLE5, because the form of (42) is very complex; for example, more complex than the form of equations used in the Fan technology [8, 9]. The Kudryashov method [11] as well as the methods [5–10, 12, 14, 16–19], proposed to search for both elliptic and elementary solutions at all values of the numeric parameters such that single-valued solutions can exist. The use of the residue theorem and the Hone method allows us to fix some of numeric parameters and to simplify calculations without loss of elliptic solutions. Note that we not only find elliptic solutions for the CGLE5, but also prove that there are no other elliptic solutions for $g_r \neq 0$. Using the standard methods one cannot prove the uniqueness of the obtained elliptic solutions.

In contrast to [5, 27], we consider a system of differential equations instead of the equivalent single differential equation and demonstrate that system (6) is more convenient for analysis than a single equation for M. Indeed, if we analyse only the Laurent-series expansions of the function M, then we should also consider the case when M has not four, but only two different Laurent-series expansions, beginning from B_1 and B_2 or B_3 and B_4. The consideration of the Laurent-series expansions of two functions: $\tilde{\psi}$ and M allows us to reject this possibility. Moreover, we can choose a function, whose analytic form we want to find. Instead of the functions $\tilde{\psi}$ and M we can consider some combination, for example, a polynomial, of these functions and seek this combination in the analytic form. Note that we have no need of any differential equation for this combination. So, we can conclude that the use of the Laurent-series solutions gives new insight into the problem of constructing of exact solutions for nonintegrable systems.

Note that the Hone method and, therefore, our approach, is very effective in the case of the CGLE5, because coefficients of the Laurent-series solutions depend only on parameters of equations, i.e. they do not include additional arbitrary parameters (have no resonances). It is an important problem to generalize the Hone method on the Laurent-series solutions with resonances.

Another way for future investigations is the generalization of the Conte–Musette method on the case of multivalued solutions. The first results in this direction have been obtained in [19, 50].

Acknowledgments

This research is supported in part by RFBR grant 05-01-00758 and by Russian Federation President's grant NSh-8122.2006.2.

Appendix. Properties of elliptic functions

The function $\varrho(z)$ of the complex variable z is a doubly periodic function if there exist two numbers ω_1 and ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$, such that for all $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z+\omega_1) = \varrho(z+\omega_2). \tag{A.1}$$

By definition a double-periodic meromorphic function is called an elliptic function [49, 51]. These periods define the period parallelograms with vertices z_0, $z_0+N_1\omega_1$, $z_0+N_2\omega_2$ and $z_0+N_1\omega_1+N_2\omega_2$, where N_1 and N_2 are arbitrary natural numbers and z_0 is an arbitrary complex number. A parallelogram of periods is called the fundamental parallelogram of periods, if it does not include other parallelograms of periods, which corresponds to $N_1 = N_2 = 1$.

The classical theorems for elliptic functions [49, 51] prove that

- If an elliptic function has no poles then it is a constant.
- The number of elliptic function poles within any finite-period parallelogram is finite.
- The sum of residues within any finite-period parallelogram is equal to zero (the residue theorem).
- If $\varrho(z)$ is an elliptic function then any rational function of $\varrho(z)$ and its derivatives is an elliptic function as well.
- For each elliptic function $\varrho(z)$ there exist an integer number m ($m \geqslant 2$) and coefficients $h_{i,j}$ such that $\varrho(z)$ is a solution of equation (2).

References

[1] Akhmediev N N and Afanasjev V V 1996 *Rev. Phys.* E **53** 1190–201
[2] Conte R and Musette M 1992 *J. Phys. A: Math. Gen.* **25** 5609–23
[3] Conte R and Musette M 1993 *Physica* D **69** 1–17
[4] Marcq Ph, Chaté H and Conte R 1994 *Physica* D **73** 305–17 (*Preprint* patt-sol/9310004)
[5] Musette M and Conte R 2003 *Physica* D **181** 70–9 (*Preprint* nlin.PS/0302051)
[6] Conte R and Musette M 2005 Solitary waves of nonlinear nonintegrable equations, chapter in the book 'Dissipative Solitons' (*Lect. Notes Phys.* vol 661) ed N N Akhmediev and A Ankiewicz (Berlin: Springer) (*Preprint* nlin.PS/0407026)
[7] Nikitin A G and Barannyk T A 2003 Solitary wave and other solutions for nonlinear heat equations *Preprint* math-ph/0303004
[8] Fan E 2002 *J. Phys. A: Math. Gen.* **35** 6853–72
[9] Fan E 2003 *J. Phys. A: Math. Gen.* **36** 7009–26
[10] Kudryashov N A 1989 *Matematicheskoye modelirovanie* **1** 151–8
[11] Kudryashov N A 2003 Nonlinear differential equations with exact solutions expressed via the Weierstrass function *Preprint* nlin.CD/0312035
[12] Kudryashov N A 2004 Simplest equation method to look for exact solutions of nonlinear differential equations *Preprint* nlin.SI/0406007
[13] Kudryashov N A and Demina M V 2006 Polygons for finding exact solutions of nonlinear differential equations *Preprint* nlin.SI/0601035
[14] Grimshaw R and Pavlov M 1999 *Phys. Lett.* A **251** 25–30
[15] Porubov A V 1993 *J. Phys. A: Math. Gen.* **26** L797–800
[16] Santos G S 1989 *J. Phys. Soc. Japan* **58** 4301–10

[17] Timoshkova E I 1999 *Astron. Rep.* **43** 406–11
Timoshkova E I 1999 *Russ. Astron. J.* **76** 470–5
[18] Vernov S Yu and Timoshkova E I 2005 *Phys. At. Nucl.* **68** 1947–55
Vernov S Yu and Timoshkova E I 2005 *Yad. Fiz.* **68** 2008–16 (*Preprint* math-ph/0402049)
[19] Vernov S Yu 2006 *J. Nonlinear Math. Phys.* **13** 50–63
[20] Vladimirov V A and Kutafina E V 2004 *Rep. Math. Phys.* **54** 261–71
[21] Kudryashov N A 2004 *Analytic Theory of Nonlinear Differential Equations* (Moscow, Izhevsk, RCD)
[22] Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715–21 and 1006–15
[23] Tabor M 1989 *Chaos and Integrability in Nonlinear Dynamics* (New York: Wiles)
[24] Hone A N W 2005 Painlevé test, singularity structure, and integrability *Preprint* nlin.SI/0502017
[25] Vernov S Yu 2003 *Theor. Math. Phys.* **135** 792–801
Vernov S Yu 2003 *Teor. Mat. Fiz.* **135** 409–19 (*Preprint* math-ph/0209063)
[26] Kudryashov N A 2004 Painlevé property and the first integrals of nonlinear differential equations *Preprint* nlin.SI/0408041
[27] Hone A N W 2005 *Physica* D **205** 292–306
[28] Vernov S Yu 2006 *Theor. Math. Phys.* **146** 131–9
Vernov S Yu 2006 *Teor. Mat. Fiz.* **146** 161–71 (*Preprint* nlin.PS/0503009)
[29] Baldwin D and Hereman W 2006 *J. Nonlinear Math. Phys.* **13** 90–110 (*Preprint* nlin.SI/0505004)
[30] Renner F 1992 *Comput. Phys. Commun.* **70** 409–16
[31] Scheen C 1997 *Theor. Comput. Sci.* **187** 87–104
[32] Xu G Q and Li Z B 2004 *Comput. Phys. Commun.* **161** 65–75
[33] Drouffe J-M 1996 Simplex AMP reference manual, version 1.0 (1996), SPhT, CEA Sacley, F-91191 Gif-sur-Yvette Cedex
[34] Vernov S Yu 2005 Packages of Procedures for Maple 9 (‘ellipso’) and REDUCE 3.7, 2005, http://theory.sinp.msu.ru/∼svernov/programs/
[35] Vernov S Yu 2004 Construction of single-valued solutions for nonintegrable systems with the help of the Painlevé test *Proc. Int. Conf. ‘Computer Algebra in Scientific Computing’ (St. Petersburg, Russia, 2004)* ed V G Ganzha, E W Mayr and E V Vorozhtsov (Munchen, Garching: Technische Universitat) pp 457–65 (*Preprint* nlin.SI/0407062)
[36] Vernov S Yu 2005 Interdependence between the Laurent-series and elliptic solutions of nonintegrable systems *Proc. Int. Conf. ‘Computer Algebra in Scientific Computing’ (Kalamata, Greece, 2005) (The Lecture Notes of Computer Science (Springer) LNCS* vol 3718*)* ed V G Ganzha, E W Mayr and E V Vorozhtsov (Berlin: Springer) pp 457–68
[37] Buchberger B 1976 *SIGSAM Bull.* **39** 19–29
Buchberger B 1976 *SIGSAM Bull.* **40** 19–24
[38] Davenport J, Siret Y and Tournier E 1987 *Calcul Formel, Systemes et Algorithmes de Manipulations Algebriques* (Paris, New York: Masson)
[39] Gerdt V P and Blinkov Yu A 1998 *Math. Comput. Simul.* **45** 519–42 (*Preprint* math.AC/9912027)
Gerdt V P and Blinkov Yu A 1998 *Math. Comput. Simul.* **45** 543–60 (*Preprint* math.AC/9912029)
[40] Gerdt V P 2005 Involutive algorithms for computing Gröbner basis *Computational Commutative and Non-commutative Algebraic Geometry (NATO Science Series)* ed S Cojocaru, G Pfister and V Ufnarovski (Bristol: Institute of Physics Publishing) pp 199–225 (*Preprint* math.AC/0501111)
[41] Eremenko A E 2005 Meromorphic traveling wave solutions of the Kuramoto–Sivashinsky equation *Preprint* nlin.SI/0504053
[42] Ginzburg V L and Landau L D 1950 *Zh. Eksp. Teor. Fiz. (Sov. Phys.—JETP)* **20** 1064–82
Landau L D 1965 *Collected Papers* (Oxford: Pergamon) p 546 (Engl. Transl.)
[43] Aranson I and Kramer L 2001 *Rev. Mod. Phys.* **74** 99–143 (*Preprint* cond-mat/0106115)
[44] van Saarloos W and Hohenberg P C 1992 *Physica* D **56** 303–67
van Saarloos W and Hohenberg P C 1993 *Physica* D **69** 209 (erratum)
[45] Cross M C and Hohenberg P C 1993 *Rev. Mod. Phys.* **65** 851–1112
[46] Doelman A 1996 *Physica* D **97** 398–428
[47] van Saarloos W and Hohenberg P C 1990 *Phys. Rev. Lett.* **64** 749–52
[48] Kengne E and Liu W M 2006 *Phys. Rev.* E **73** 026603
[49] von Hurwitz A 1964 *Allgemeine Funktionentheorie und Elliptische Funktionen, von R. Courant, Geometrische Funktionentheorie* (Berlin: Springer)
[50] Vernov S Yu 2006 *Programming and Comput. Software* **32** 77–83
[51] Erdélyi A *et al* 1955 *Higher Transcendental Functions (based, in part, on notes left by H Bateman)* vol 3 (New York: Mc Graw-Hill)